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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY.

A PQ *PLANE FOR THERMODYNAMIC CYCLIC ANALYSIS.*

By Harvey N. Davis.

With Three Charts and Twenty-one Figures.

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A *PQ* PLANE FOR THERMODYNAMIC CYCLIC ANALYSIS.

BY HARVEY N. DAVIS.

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FOUR years ago, Dr. C. E. Lucke pointed out in an admirable paper * the value of what he calls a "cyclic analysis of heat engines." The variety of processes which are available when the working substance is a gas rather than a vaporized liquid is so great, and the influence of the nature and dimensions of a given cycle, not only upon its efficiency, but upon many other properties, is so complicated, that it is evidently both interesting and important to make a purely theoretical study of cycles as such, and to endeavor to obtain statements of the questions involved in terms of the cycles themselves. And for this purpose, the best methods will almost always be graphical ones, as these are at once the most powerful in investigation and the clearest in exposition. It is desired, in this paper, to reiterate Dr. Lucke's insistence on the value of work of this kind, and to develop a method which shall be more general, and therefore more fruitful, than the one-dimensional graphics which he has proposed.

It is best to begin by inquiring how much must be given to determine completely a cycle of a given type. If the cycle is to be represented on the ordinary $p v$ plane of thermodynamics, both its dimensions and its position in the plane must be known. To describe its position will always require two coördinates, and it will be convenient to take for these the pressure (P_a) and the temperature (T_a) of the working substance at the beginning of the cycle, since these quantities are easily measured; often they are simply the pressure and temperature of the earth's atmosphere at the time, and so are not controllable, and for this reason it is best to regard them, not as variables, but as parameters in the equa-

* "A Method of Cyclic Analysis of Heat Engines," by Charles E. Lucke. The School of Mines Quarterly, **22** [1901], pp. 223, 329, 411.

tions in which they occur, and to assume for them constant "standard" values. The dependence of the results upon these values will then be a separate, and usually a very simple problem. For the determination of the dimensions of a cycle two more coördinates are usually needed, which may be thought of as describing in some generalized way the height and breadth of the cycle. Thus for a Carnot cycle the temperature and entropy ranges might be used, or the pressure or volume ranges along two adjacent sides, or some less obvious coördinates, as, for instance, the quantities of heat involved in two successive transformations. It is hoped that this paper will show that in a very great number of cases it is advantageous to choose as these two coördinates the ratio of compression (p_b/p_a), which will be denoted by P, and the heat taken in from outside sources by a unit mass of working substance during its passage once around the cycle, which will be denoted by Q. The chief reasons for this choice are to be found in the simplicity of the formulae to which it leads, and in the ease with which the physical interpretation of these formulae can be brought out. It may also be noticed that these quantities are simply and independently controllable and are easily measured.

If values of Q be taken as abscissae and values of P as ordinates, the result is a PQ plane upon which a point represents and completely describes a thermodynamic cycle (when its type and the values of the parameters are known) in exactly the same way that a point on the pv plane represents and completely describes a thermodynamic state (when the form of the characteristic equation is known, together with the values of the constants which it involves). Any property (X) of such a cycle can then be plotted along a third rectangular axis, and the resulting three-dimensional surface will give a complete picture of that property for all cycles of the type under consideration. If such a surface be constructed for each different cyclic type to be studied, one can read off from this set of models any desired information whatever about the property to which they correspond. The simplest way to handle such surfaces is to map their contour lines ($X =$ a constant) on the basal PQ plane, just exactly as isothermals or adiabatics are mapped on the familiar pv plane of thermodynamic states.

We have said that two coördinates will usually be needed to determine the dimensions of a cycle, but in many exceptional cases one is enough. An example is a cycle which consists of an isothermal, an isopiestic, and an adiabatic (Figure 1). The point a, when it is fixed by the parameters, determines the isothermal and the adiabatic upon which

it lies, and then a statement of (say) the specific volume at b settles everything. So does a knowledge of P ($\equiv p_b/p_a$). So also does a knowledge of Q (that is, the heat that must be supplied to the working substance per unit mass to carry it from b to c). In such cases, there exists a relation, $f(P, Q) = 0$, by means of which either of these quantities can be expressed in terms of the other (and the parameters). On the $P\,Q$ plane this equation is a curve; and the graphical statement of these facts is that it is impossible to construct a cycle of the given type corresponding to any point of the $P\,Q$ plane which is not also a point of this curve. Such cycles (which may be called cycles of the first order) fill, not a whole $P\,Q$ plane, but only a line in such a plane, and the

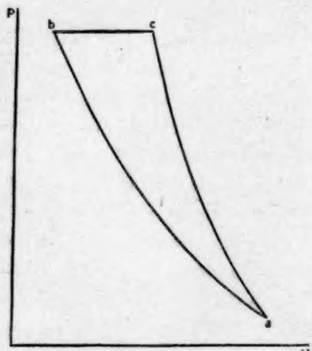

FIGURE 1.

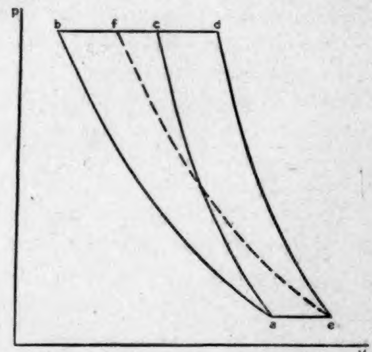

FIGURE 2.

$P\,Q\,X$ surface is defined only along this line, and degenerates into a curve drawn upon the surface of a cylinder erected on the line as a directrix. In such special cases, it is sometimes best to discard entirely the contour-line method of studying the surface, and to use instead either a development of the cylinder, or some picture of it, for example such an orthographic projection as will eliminate P. This last is exactly what Dr. Lucke did (although from a different point of view) whenever he had a first order cycle to deal with.* The reason that some such

* Of the cycles of pp. 225–234 of his paper, Numbers I, IC, VI and VIII are of the first order (Cycle VIII being a special case of the cycle discussed above); and the lines marked I and IC in the figures of pp. 418–429 are orthographic projections of the kind described. All the other lines in these figures may be thought of as the intersections of a $P\,Q\,X$ surface with a plane perpendicular to the $P\,Q$ plane along either the line $P = 2$ or the line $P = 10$.

proceeding is not always necessary is that cycles of the first order very often may be thought of simply as those special cases of second order cycles which happen to lie on certain curves in the $P\ Q$ plane. Thus a complete representation of the properties of the cycle of Figure 1 will turn up, quite incidentally, in the last part of this paper.

Perhaps the best way to bring out the scope and power of the graphical method proposed will be to carry through, in some considerable detail, an application of it to a definite problem; an excellent problem for this purpose is presented by the cycles which come up in connection with such turbines as use air instead of steam as the working substance. The simplest case would be that of a turbine of the De Laval type, in which the working substance expands adiabatically from a high pressure to the pressure of the exhaust in a single nozzle so shaped that the available energy in the working substance appears as kinetic energy. The momentum of the moving stream is then used to turn a wheel. The sequence of processes in such a machine * is as follows:

1. Air is taken in at atmospheric pressure and temperature (corresponding to the point *a* of Figure 2) and compressed.

2. It is then heated; this will be done at constant pressure, for the pressure cannot increase if a continuous flow is to be kept up, and if it diminishes during the heating, energy is needlessly lost. (This process is represented by the horizontal line ending at *d*.)

3. There is adiabatic expansion in a nozzle (line *d e*).

4. Either the hot exhaust (point *e*) is thrown away at atmospheric pressure and new air is taken in, or the same working substance is cooled and used over again. (Either process is represented by the line *e a*.)

The only thing not yet described is the nature of the compression, and this may be adiabatic (line *a c*) or isothermal (line *a b*) or anything between. The two simplest cases may be taken as typical; the corresponding cycles will be called

Type A. The Adiabatic or Brayton Cycle (a c d e a of Figure 2).

Type B. The Isothermal Cycle (a b d e a of Figure 2).

Of the non-atmospheric cycles omitted from the list on p. 417 as being not "accurately defined," all but cycle V (of p. 233) are of the third order; such cycles would require three variables and a three-dimensional graphics, which would bear to the $P\ Q$ plane exactly the same relation which that plane bears to the Q axis of Dr. Lucke's paper. Luckily very few important cycles are of higher order than the second.

* Here, as elsewhere in this paper, only purely ideal conditions are considered.

In a cycle of type A, the temperature of the working substance after compression (point *c*) is already comparatively high, but in the second cycle this temperature (point *b*) is still that of the atmosphere, and therefore, in general, much lower than that of the exhaust (point *e*); the second cycle, therefore, but not in general the first, may be modified by introducing the idea of regeneration. If *ef* be an isothermal through *e*, the working substance may be carried from the state *b* to the state *f* by means of the heat which the exhaust gives up in passing from the state *e* to the state *a*, and this too without any loss of available energy, by simply passing the high pressure air and the exhaust in opposite directions through adjacent passages. A third type of cycle is thus obtained, and may be called

Type C. The Regenerative Cycle (*a b f d e a of Figure* 2).

Since these cycles themselves and not their applications to engineering are to be studied in this paper, there is no need of ruling out those cases under types *B* and *C* in which the point *e* falls to the left of the point *a*. Such cycles are figures-of-eight, and the work done by the working substance may perfectly well be zero, or less than zero, but the properties of such degenerate cycles fit on continuously with those of cycles of the ordinary kind, and should be studied with them. In a figure-of-eight case under type *C* the regenerative process consists in *cooling* the high pressure air to the temperature of the exhaust, and cycles of this kind are of some interest in connection with liquid air machines.

In the study of gas turbine cycles a number of properties should be considered. The most important of these is the temperature (T_e) of the working substance after its expansion (point *e*), for if this be too high, the blades of the turbine wheel will suffer. Another important property is the velocity (V) of the stream of gas as it leaves the nozzle, and here also there is an upper limit, in that V should bear a definite ratio to the peripheral velocity of the turbine wheel, and this is always limited by structural or other considerations. There is also the efficiency (E) of the cycle; and finally the work (W) which a unit mass of the working substance does in going once around a cycle is of interest because of its bearing on the size of the machinery necessary for a given output. Twelve sets of contour lines are therefore involved, four for each type of cycle; and the proposed problem is to determine by means of them (1) which type of cycle is preferable, and (2) what the dimensions of a cycle of that type should be, it being desirable to make E and W as large as possible subject to the conditions $T_e \leqq T_e$ and $V \leqq V$, where T_e and V are given constants.

In obtaining the formulae upon which the following discussion is based, it will be assumed that the working substance is a perfect gas. The analytic expression of this assumption is the two equations

$$p v = R T,$$

where R is a constant, and

$$\frac{C_p}{C_v} = \kappa,$$

where κ is a constant.* Of the four constants R, C_p, C_v, and κ, two must be given to determine the nature of the perfect gas; the two which will be used in the formulae of this paper are C_p and κ. The only other symbols which will be needed are the physical constants g and J, the parameters p_a and T_a, the variables P and Q, and the symbol (T_e, V, E or W) which represents the property under consideration. For convenience of reference the formulae are collected in Table 1; they are easily obtained by the ordinary methods.† It should be noticed that not one of them happens to involve the parameter p_a, a simplification which has resulted from the choice of P as a coordinate. Each of these formulae (except the E-type-A one) can be explicitly solved for Q. In the accompanying figures, values of Q have been calculated with considerable care, usually for some twenty-five different values of P, and it is hoped that the resulting curves, especially on the working charts, are

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* The gas law alone is not enough; if both are given it follows at once that C_p and C_v are themselves constants.

† The T_e formulae for types A and B were obtained by following pressures and temperatures up the compression line, across the top, and down again, by means of the usual equations. T_e for the regenerative type is independent of the parameters. The work was as follows (see Figure 2):

$$T_f = T_e\,; \quad \therefore \quad T_d = T_e + \frac{Q}{C_p}. \qquad \text{(a)}$$

But the equation of the adiabatic $d\,e$ gives

$$T_d\, p_d^{\frac{1-\kappa}{\kappa}} = T_e\, p_e^{\frac{1-\kappa}{\kappa}}, \quad \text{or} \quad T_d = T_e\, P^{\frac{\kappa-1}{\kappa}}; \qquad \text{(b)}$$

therefore, etc. The V equations are special cases of equation (208), p. 151, of Professor Peabody's "Thermodynamics of the Steam Engine" [4th ed. 1900]. The E equations were obtained from the amounts of heat involved in the various transformations, and the W equations are of the form $W = J Q E$.

TABLE 1.

		Type A. Brayton.	Type B. Isothermal.	Type C. Regenerative.
Final Temperature.	T_e	$\frac{Q}{C_p}\frac{1}{P^{\frac{\kappa-1}{\kappa}}} + T_a$	$\left(\frac{Q}{C_p} + T_a\right)\frac{1}{P^{\frac{\kappa-1}{\kappa}}}$	$\frac{Q}{C_p}\frac{1}{P^{\frac{\kappa-1}{\kappa}} - 1}$
Velocity.	V	$\sqrt{2gJ(Q + C_p T_a P^{\frac{\kappa-1}{\kappa}})(1 - P^{\frac{1-\kappa}{\kappa}})}$	$\sqrt{2gJ(Q + C_p T_a)(1 - P^{\frac{1-\kappa}{\kappa}})}$	$\sqrt{2gJQ}$
Efficiency.	E	$1 - P^{\frac{1-\kappa}{\kappa}}$	$\frac{(Q + C_p T_a)(1 - P^{\frac{1-\kappa}{\kappa}}) - C_p T_a \log_e P^{\frac{\kappa-1}{\kappa}}}{Q}$	$\frac{Q - C_p T_a \log_e P^{\frac{\kappa-1}{\kappa}}}{Q}$
Work.	W	$JQ(1 - P^{\frac{1-\kappa}{\kappa}})$	$J(Q + C_p T_a)(1 - P^{\frac{1-\kappa}{\kappa}}) - JC_p T_a \log_e P^{\frac{\kappa-1}{\kappa}}$	$JQ - JC_p T_a \log_e P^{\frac{\kappa-1}{\kappa}}$

sufficiently accurate for any purpose for which such graphs could reasonably be used.*

Type A. The Brayton Cycle. The final temperature curves for this type are represented in Figure 3, and their general shape shows at once

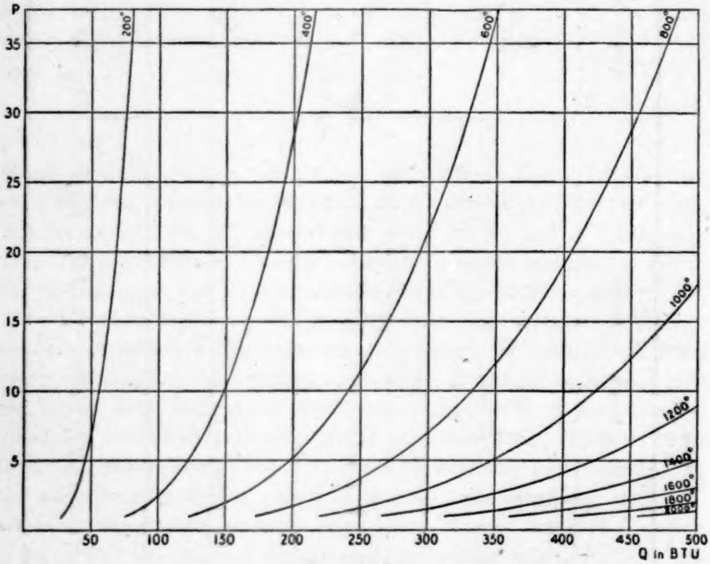

FIGURE 3.

Type A. Final temperature. Showing, for Brayton cycles (adiabatic compression), the temperature of the gas after its expansion.

the advantage of a high compression ratio. It should be noticed that any final temperature not lower than the initial temperature is possible, it being only necessary to choose a sufficiently small value for Q. That

* In the numerical work, the following constants (mostly from Peabody) were used:

$$C_p = 0.2375 \text{ B. T. U.},$$
$$\kappa = 1.405,$$
$$g = 32.17 \text{ feet per second},$$
$$J = 778.$$

The standard value for the temperature parameter was taken as

$$t_a = 80° \text{ F.} \qquad \text{or} \qquad T_a = 541° \text{ F. Abs.}$$

The unit of mass used was one pound (gravitational). The resulting T_e is in Fahrenheit degrees absolute, but the results have always been expressed on the ordinary scale. V is in feet per second, and W in foot pounds.

curve of the family for which $T_e = T_a$ coincides with the P axis. All the curves end in the line $P = 1$.

The velocity curves are in Figure 4. Any velocity greater than zero is possible, but here P, as well as Q, must be taken sufficiently small. All the V curves start from finite points on the P axis but run off asymptotically to the line $P = 1$. The velocities in the central parts of the figure are of the same order as those obtainable with steam.

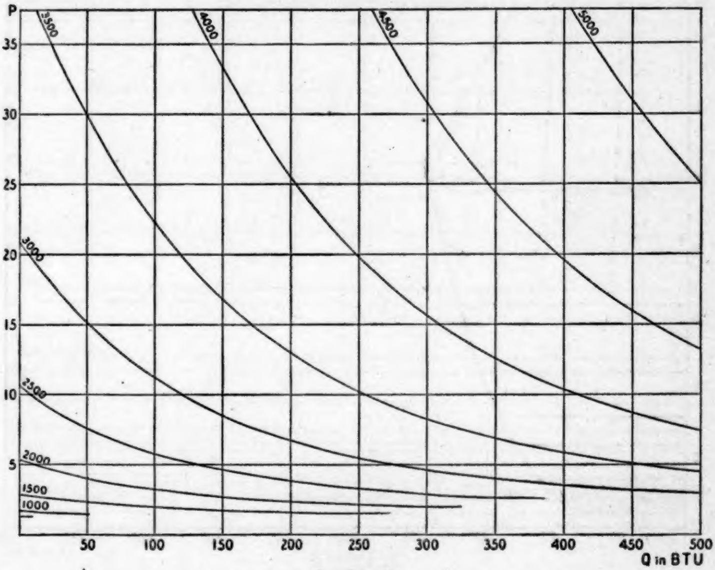

FIGURE 4.

Type A. Velocity. Showing, for Brayton cycles, the velocity in feet per second of the stream of gas as it strikes the blades of the turbine.

In the problem proposed, upper limits were assigned both to T_e and to V, and we have called these limits $\bar{T}_e$ and $\bar{V}$. That is, speaking graphically, no cycle is available whose point lies beyond either of the curves $T_e = \bar{T}_e$ or $V = \bar{V}$, and the remote parts of the $P\,Q$ plane are to be left out of consideration. There remains a finite portion of the plane (the shaded area of Figure 5*) within which an available cycle must lie. It may be called the $\bar{V}\,\bar{T}_e$ area, or, less explicitly, a $V\,T$ area.

* The limits represented in Figure 5 are $\bar{t}_e = 400°\ F.$ (that is $\bar{T}_e = 861°\ F.$ Abs.) and $\bar{V} = 4000$ ft. per sec.

The second part of the problem proposed for this discussion is equivalent to the question, which point in a VT area is best for the purposes in hand. The answer is given by the other two sets of curves.

The E curves are shown in Figure 6, and are straight lines, the efficiency of a cycle of this type being independent of its breadth. Figure 7 shows the W curves. The line $P = 1$ is a part of the curve $W = 0$ because it is also the curve $E = 0$. The P axis is also a part

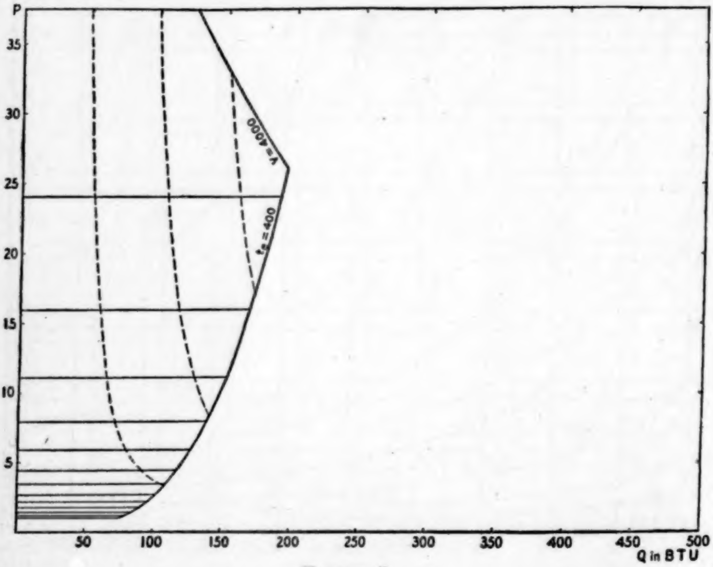

FIGURE 5.

Type A. VT area. This area (of which only the lower part is included within the limits of the figure) contains all the adiabatic cycles which satisfy the conditions $T_e \leqq \bar{T}_e$ and $V \leqq \bar{V}$. The curves within give the efficiency and work of these available cycles and enable one to choose between them.

of the curve $W = 0$, because a type A cycle whose point is on that axis, has zero area. Each of the other curves has these lines as asymptotes.

In Figure 5 the full lines are efficiency curves, and the dotted lines are work curves. If efficiency is the important thing, the best point is evidently as far up in the area as possible, that is at the intersection of the curve $V = \bar{V}$ with the P axis. The corresponding cycle is a null cycle ($Q = W = 0$) and must be regarded as a limiting case in the direction of increasing efficiency. If, on the other hand, the important

thing is to decrease the amount of air handled, as will often be the case,* then the best point in the area is the intersection of the curve $V = \bar{V}$ with the curve $T_e = \bar{T}_e$. Under such circumstances there is a perfectly definite and comparatively low ratio of compression which should not be exceeded. The intersection of the curves $V = \bar{V}$ and $T_e = \bar{T}_e$ may be called for convenience the $\bar{V}\bar{T}_e$ corner, or, less explicitly, a VT corner.

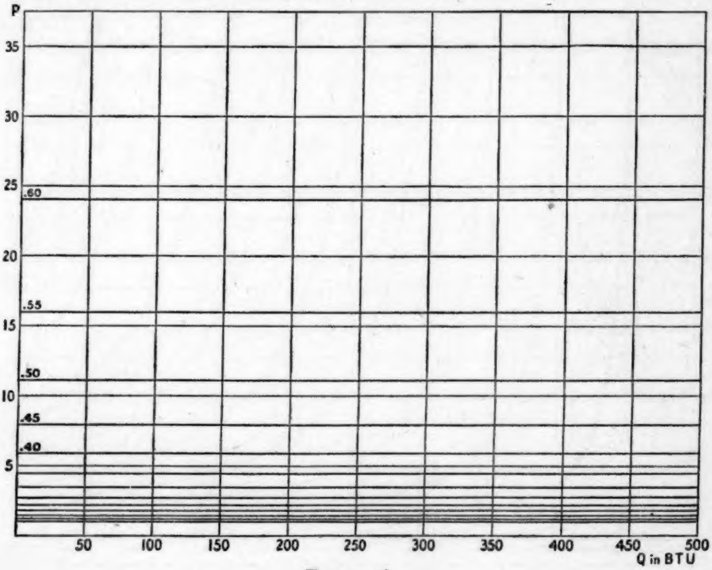

FIGURE 6.

Type A. Efficiency. The efficiency of a Brayton cycle is independent of its breadth.

All four of these families of curves are plotted together for reference on the first working chart, the T_e and V curves as lines, the E curves in dots, and the W curves in dashes. Of course the result is a complicated plane; but it is believed that a study of the figures in the text will make this chart intelligible and that the information of various kinds which it can afford will repay one for this trouble. To avoid unnecessary confusion, the efficiency lines between $E = .00$ and $E = .25$

* Both the size of the necessary machinery and the inefficiency of air compressors point that way. Highly compressed air is sometimes a more expensive commodity than fuel.

have been omitted from this and the two following charts, as being unimportant.

Type B. The Isothermal Cycle. The T_e and V curves for this type are shown in Figures 8 and 9. They are the corresponding curves for the adiabatic case, sheared horizontally in such a way that the $T_e = T_a$ curve, which formerly lay along the P axis, has now a position indicated by the dotted line of Figure 8.* All cycles whose points lie on this line

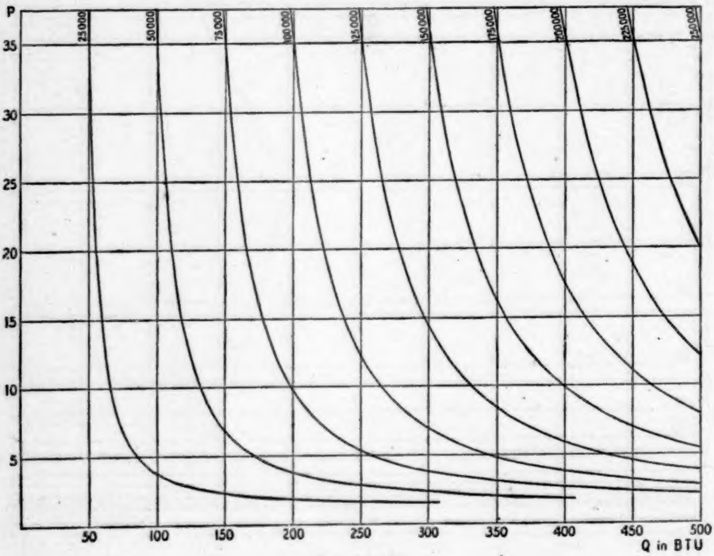

FIGURE 7.

Type A. Work. These curves give the net work in foot pounds which one pound of gas can do while making one complete circuit of a Brayton cycle.

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* For if the T_e formulae (types A and B) be solved for Q, they give

$$ {}_{T_e}Q_A = C_p\,(T_e - T_a)\,P^{\frac{\kappa-1}{\kappa}}, \qquad \text{(a)} $$

where ${}_{T_e}Q_A$ stands for the Q of any point of a T_e curve for the type A case; also

$$ \begin{aligned} {}_{T_e}Q_B &= C_p\,(T_e\,P^{\frac{\kappa-1}{\kappa}} - T_a) \qquad \text{(b)} \\ &= C_p\,(T_e - T_a)\,P^{\frac{\kappa-1}{\kappa}} + C_p\,(T_a\,P^{\frac{\kappa-1}{\kappa}} - T_a) \\ &= {}_{T_e}Q_A + ({}_{T_e}Q_B)_{T_e = T_a}. \end{aligned} $$

are peculiar in that for them the point e of Figure 2 coincides with the point a of that figure; in other words this line $(T_e = T_a)$ is a locus of first order cycles of the type of Figure 1, which thus appear as special cases of the second order cycles of type B. It is evident that the abscissa of any point of this locus represents the amount of heat that must be supplied to a unit mass of isothermally compressed air, to raise its temperature to that of adiabatically compressed air at the same pressure;

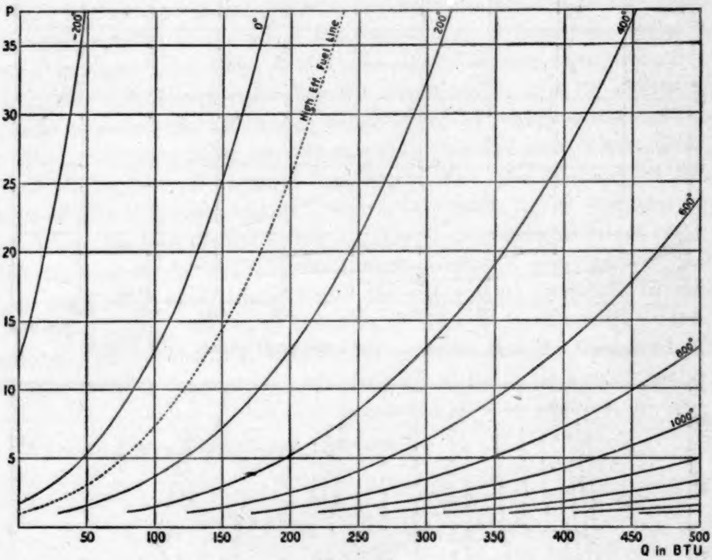

FIGURE 8.

Type B. Final temperature. For cycles with isothermal compression. (Cf. Figure 3.)

and therefore, to bring such air to a given thermodynamic state requires an amount of heat which is the sum of this preliminary Q' and the Q_A of the corresponding adiabatic cycle. This explains the shearing of the adiabatic plane. Points to the left of this locus represent figure-of-eight cycles, so that, for the engineer, this line is a sort of natural boundary for the isothermal plane.

The VT area for the same values of $\bar{V}$ and $\bar{T}_e$ as were used before, now takes the form indicated in Figure 10. Its boundaries have simply been subjected to the horizontal shear already described, so that the P

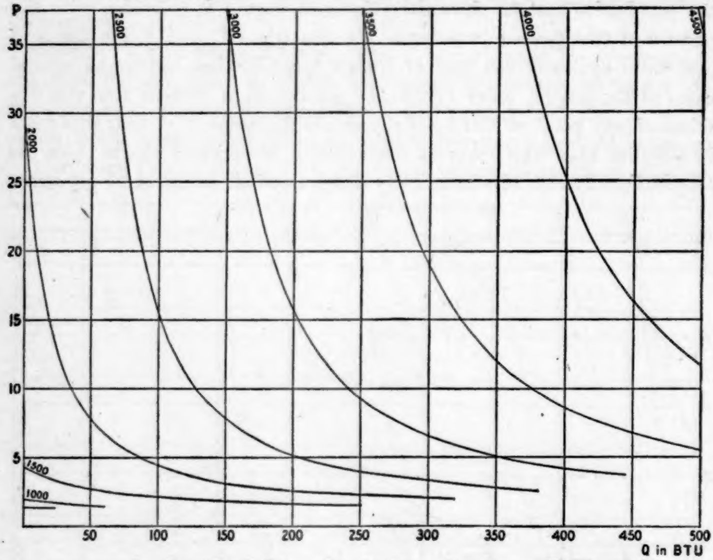

FIGURE 9. *Type B. Velocity.* For isothermal cycles. (Cf. Figure 4.)

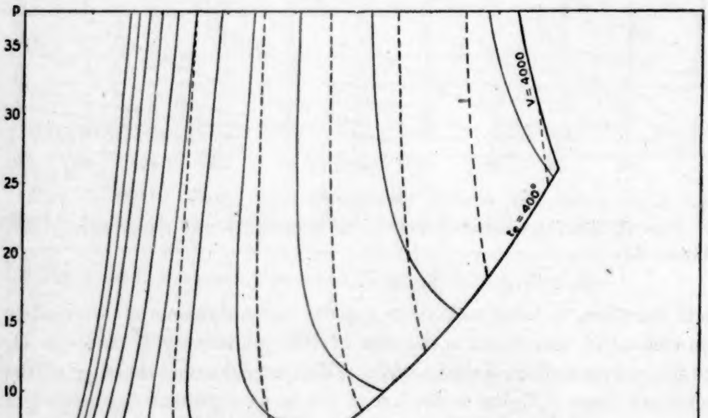

FIGURE 10. *Type B. VT area.* For isothermal cycles. (Cf. Figure 5.)

of the VT corner is the same as before; but the lines within are entirely changed.

The efficiency lines for type B are shown in Figure 11. The line $P = 1$ is still a part of the curve $E = 0$, but the other part of this curve is now well out in the plane. It is the locus of those figure-of-eight cycles whose (algebraic) area is zero. Cycles to the left of it have negative areas, and any negative efficiency whatever is possible for any value of P ($P > 1$). An extraordinary property of the curves of this family is that any vertical line cuts each of them twice if at all; their turning points lie on the above mentioned locus of first order cycles.* As P increases indefinitely, Q also increases indefinitely, logarithmically. The other end of each curve has the corresponding type A efficiency line as an asymptote. This may be explained by regarding an isothermal cycle as made up of an adiabatic cycle together with a first order cycle of the type of Figure 1 (see Figure 2). The efficiency of the type A part is independent of Q and is very much higher than that of the triangular part that goes with it. Therefore, for small values of Q, the efficiency of the whole cycle is small, because a great part of the heat supplied is made use of under the unfavorable conditions of the first order cycle. But as Q increases, P being kept constant, this low efficiency part of the heat supplied becomes less and less important in comparison with the rest, and the efficiency of the cycle approaches as

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* For the E-type-B formula can be written

$$_EQ_B = C_p\, T_a \left(\frac{\log_e P^{\frac{\kappa-1}{\kappa}} - 1 + P^{\frac{1-\kappa}{\kappa}}}{1 - P^{\frac{1-\kappa}{\kappa}} - E} \right). \qquad \text{(a)}$$

Differentiating and eliminating E, gives as the derivative

$$\frac{\partial Q}{\partial P} = \frac{\kappa - 1}{\kappa} \frac{Q}{P} \frac{1 - P^{\frac{1-\kappa}{\kappa}} - P^{\frac{1-\kappa}{\kappa}} \dfrac{Q}{C_p T_a}}{\log_e P^{\frac{\kappa-1}{\kappa}} - 1 + P^{\frac{1-\kappa}{\kappa}}}. \qquad \text{(b)}$$

The condition that this vanish is

$$1 - P^{\frac{1-\kappa}{\kappa}} - P^{\frac{1-\kappa}{\kappa}} \frac{Q}{C_p T_a} = 0.$$

or

$Q = C_p\, T_a (P^{\frac{\kappa-1}{\kappa}} - 1) = C_p\, (T_c P^{\frac{\kappa-1}{\kappa}} - T_a) = ({}_{T_c}Q_B)_{T_c = T_a}$ (see eq. b of note on p. 640).

a limit that of an adiabatic cycle with the same ratio of compression.* From this point of view, the locus of these first order cycles may be called the "high efficiency fuel line" as it indicates the beginning of the high efficiency fuel for a given value of P.

The work curves for this type are shown in Figure 12, and they, like the efficiency curves, have turning points on the high efficiency fuel line. The curve $W = 0$ is, as always, the same as the curve $E = 0$. The other lines go off to the right with $P = 1$ as an asymptote.

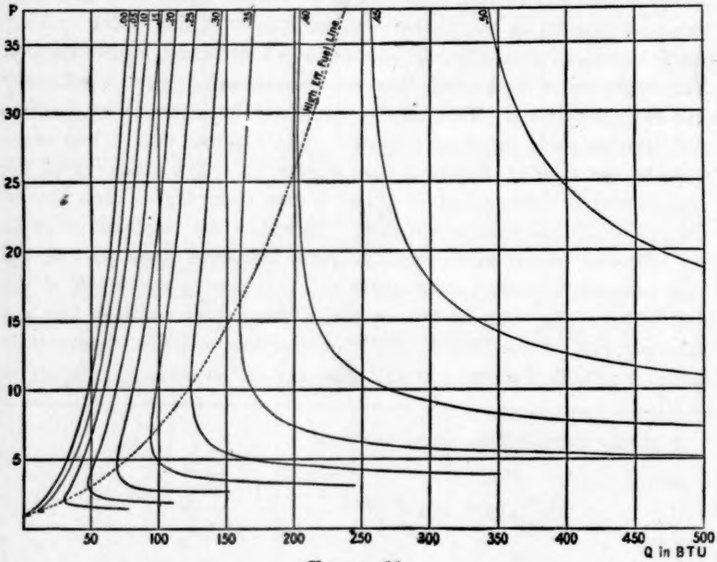

FIGURE 11.

Type B. Efficiency. For isothermal cycles. The turning points of these curves lie on the "high efficiency fuel line." (Cf. Figure 6.)

An inspection of Figure 10 shows that, as far as efficiency is concerned, it pays, in this particular case, to keep at least as far up the right hand

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* The formula can be put in the form

$$E_B = (1 - P^{\frac{1-\kappa}{\kappa}}) - \frac{C_p T_a}{Q} (\log_e P^{\frac{\kappa-1}{\kappa}} - 1 + P^{\frac{1-\kappa}{\kappa}})$$

$$= E_A - \frac{\phi(P)}{Q},$$

where $$\lim_{Q=\infty} \frac{\phi(P)}{Q} = 0.$$

boundary of the VT area as the top of the figure. In general, an efficiency curve comes in along the Q axis more horizontally than the V curves, and if followed upward far enough, swings completely back to $Q = \infty$ again, cutting each V curve twice if at all. There will always be one efficiency curve that just touches a given velocity curve. Call the point of tangency S. Its cycle has evidently a higher efficiency than any other cycle giving the same final velocity, for the efficiency decreases as a point moves away from S in either direction along the given V curve. Therefore, if the point S of the curve $V = \bar{V}$ lies on the boundary of a given $\bar{V}\,\bar{T}_e$ area, it is the best point of the area for efficiency. But if $\bar{T}_e$ is so small as to bring the corresponding $\bar{V}\,\bar{T}_e$ corner farther up the $\bar{V}$ curve than the point S for that curve, then the $\bar{V}\,\bar{T}_e$ corner itself is the best point of the area for efficiency, as it lies nearest S. The locus of the point S for all possible values of $\bar{V}$ is the line drawn with long dashes in Figure 13. It looks somewhat like one of the T_e family, but its equation is quite different.*

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* The analytic work is as follows. The equation of the V curves is

$$ {}_V Q_B = \frac{V^2}{2gJ\left(1 - P^{\frac{1-\kappa}{\kappa}}\right)} - C_p T_a. $$

Differentiating and eliminating V, gives as their slope

$$ \frac{\partial Q_V}{\partial P} = -\frac{\kappa - 1}{\kappa}\,\frac{1}{P}\,\frac{Q + C_p T_a}{P^{\frac{\kappa-1}{\kappa}} - 1}. $$

The slope of the E curves has been found (note on p. 643) to be

$$ \frac{\partial Q_E}{\partial P} = \frac{\kappa - 1}{\kappa}\,\frac{Q}{P}\,\frac{1 - P^{\frac{1-\kappa}{\kappa}} - P^{\frac{1-\kappa}{\kappa}}\dfrac{Q}{C_p T_a}}{\log_e P^{\frac{\kappa-1}{\kappa}} - 1 + P^{\frac{1-\kappa}{\kappa}}}. $$

The necessary and sufficient condition for a point of tangency is $\dfrac{\partial Q_V}{\partial P} = \dfrac{\partial Q_E}{\partial P}$, and this can be put in the form

$$ \left(1 - P^{\frac{1-\kappa}{\kappa}}\right)\left(\frac{Q}{C_p T_a}\right)^2 - \left(\log_e P^{\frac{\kappa-1}{\kappa}} + P^{\frac{\kappa-1}{\kappa}} - 3 + 2P^{\frac{1-\kappa}{\kappa}}\right)\left(\frac{Q}{C_p T_a}\right) $$
$$ - \left(\log_e P^{\frac{\kappa-1}{\kappa}} - 1 + P^{\frac{1-\kappa}{\kappa}}\right) = 0, $$

that is

$$ F_1(P)\,x^2 - F_2(P)\,x - F_3(P) = 0. $$

where $x \equiv \dfrac{Q}{C_p T_a}$. This is the equation of the desired locus. Each of the functions F is positive for values of P greater than 1, so that, of the two real roots of the equation, one is always less than zero (by Descartes' rule of signs). The other root gives the line plotted in Figure 13.

If, on the other hand, it is desirable to make W rather than E as great as possible, the best point of a VT area is always, as in the case of the Brayton cycle, the VT corner, and the best ratio of compression is the same as before. In the case of each of these types it is necessary to choose between efficiency and work, but the two best cycles are now much nearer together than they were before.

The type B working chart is similar to that for type A, except for

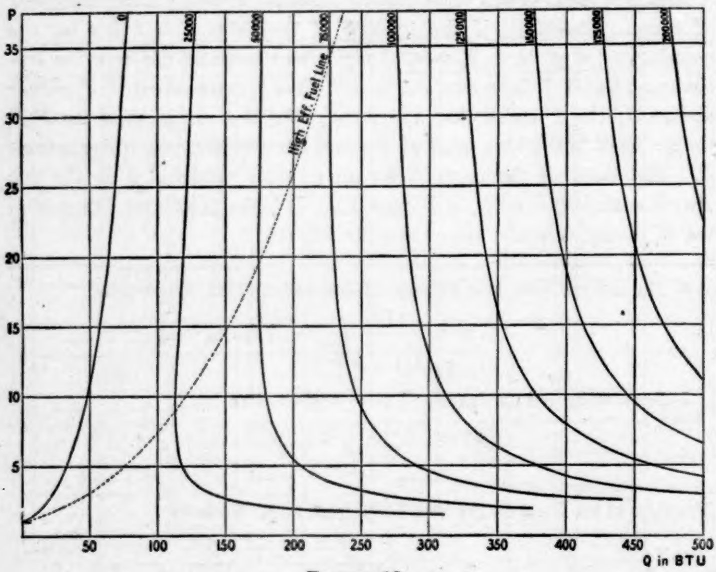


FIGURE 12.

Type B. Work. For isothermal cycles. These curves also have turning points on the "high efficiency fuel line." (Cf. Figure 7.)

the presence of the high efficiency fuel line and the locus of points S which are drawn with long dashes.

Type C. The Regenerative Cycle. The final temperature curves for type C are shown in Figure 14; they are the T_e curves for type B displaced horizontally but without distortion, until their lower ends, which were scattered along the line $P = 1$, are all brought to the single point $P = 1$, $Q = 0$. This is because T_e is (by definition) constant along one of these curves, and so, T_a being fixed, the heat $C_p(T_e - T_a)$ that can be obtained from the exhaust is the same for all cycles on that curve; consequently the Q of each point of the curve as drawn on the type B

plane will be diminished by the same amount. To the point $P = 1$, $Q = 0$ (and indeed to the whole of the lines $P = 1$ and $Q = 0$) no physical meaning can be attached; and any final temperature from absolute zero to infinity may be found in any neighborhood of this singular point.

The V curves of Figure 15 are much simpler than those for the other two types, the final velocity depending only on the "quality of the

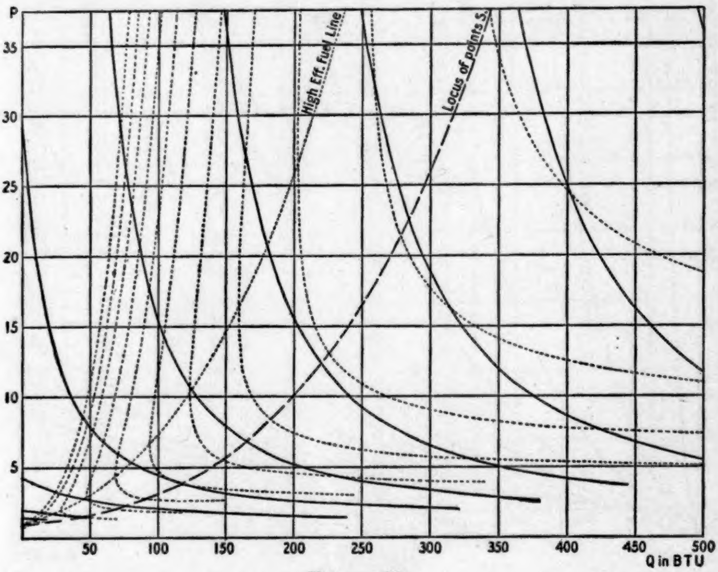


FIGURE 18.

Type B. Locus of points S. The full lines are velocity curves and the dotted lines are efficiency curves. The points of tangency lie on the locus indicated.

mixture." The E curves are represented in Figure 17, except that portions of all of them near their common intersection at the singular point have been omitted. Any efficiency from $-\infty$ to $+1$ is possible in any neighborhood of this point. The W curves (Figure 18) are all alike, and are the $E = 0$ curve indefinitely repeated at constant intervals. The $\bar{V}\bar{T}_e$ area (Figure 16) differs from those for the other types chiefly in that it extends upward indefinitely. Fortunately the VT corner is always the best point in the area, not only for work, but for efficiency as well.

A glance at the working chart for this type shows that the whole type B plane has, as it were, shrunk horizontally toward its high efficiency fuel line. That line has many of the same interesting properties on the type C plane that it had before, in that here also it is the locus of the same set of first order cycles, and is, for the engineer, a natural boundary. It is interesting to notice that any line of any family of curves whatever crosses this high efficiency fuel line at exactly the same point

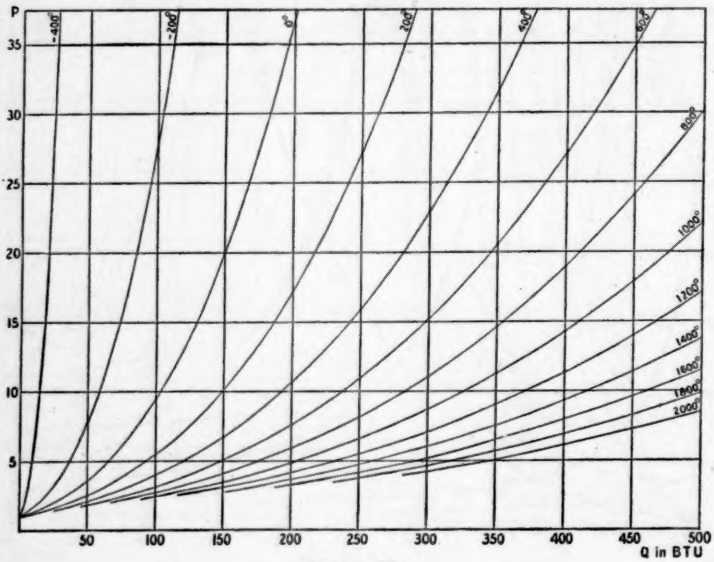


FIGURE 14.

Type C. Final temperature. These curves are for cycles with isothermal compression and a subsequent transfer of heat from the exhaust. They all start from the point $Q = 0$, $P = 1$, and there are an infinite number of them in the lower part of the plane. (Cf. Figures 3 and 8.)

in both the type B and the type C planes, although the general character of the line may be totally altered.

As has already been suggested, the dependence of the properties of these cycles upon the values of the parameters p_a and T_a is very simple. In no case is p_a involved at all. The efficiency and work of a Brayton cycle and the final temperature and velocity of a regenerative cycle are independent of T_a also, and the curves of Figures 6, 7, 14, and 15 may be used whatever the initial conditions. In the isothermal case, the T_e and

V equations, when solved for Q, involve T_a in an additive constant, so that the effect of changing it is to shift the whole $P\,Q$ plane along the Q axis without distortion; the curves of Figures 8 and 9 may be used for the general case provided only that all readings of Q are increased or decreased by a suitable constant. The efficiency equations for types B and C, when solved for Q, involve T_a as a factor, so that the curves of Figures 11 and 17 can always be used by multiplying values of Q by a

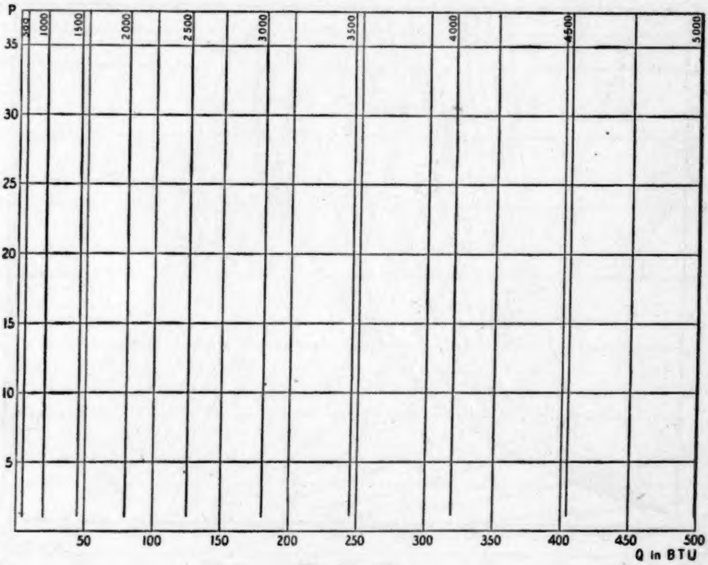


FIGURE 15.

Type C. Velocity. The velocity which a regenerative cycle produces depends only on the "quality of the mixture." (Cf. Figures 4 and 9.)

suitable constant. And in the other cases, the dependence upon T_a, while slightly less simple, can be easily followed out. None of the re-

TABLE 2.

	Brayton.				Isothermal.				Regenerative.			
	p_a	T_a	P	Q	p_a	T_a	P	Q	p_a	T_a	P	Q
T_e		•	•	•		•	•	•			•	•
V		•	•	•		•	•	•				•
E			•			•	•	•		•	•	•
W			•	•		•	•	•		•	•	•

sults of this paper are essentially affected by a change in T_a. The dependence of the principal properties upon the four coördinates is summarized in Table 2, a blank space indicating that the coördinate at the top of the column is not involved in the corresponding equation.

The second of the questions proposed at the beginning of this investigation has now been answered for each of the three cyclic types; the

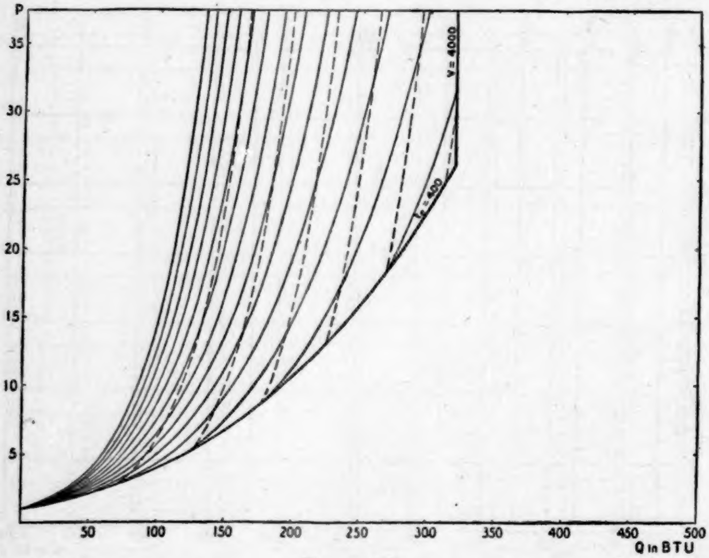


FIGURE 16.

Type C. VT area. For regenerative cycles. (Cf. Figures 5 and 10.)

TABLE 3. THE BEST POINTS.

	For Efficiency.	For Work.
Type A. Brayton . .	The top of the VT area, on the P axis (a null cycle).	The VT corner.
Type B. Isothermal .	The point S.	The VT corner.
Type C. Regenerative	The VT corner.	The VT corner.

first question involves simply a comparison of the cycles thus selected. These are distributed in the VT area as is indicated in Table 3, and their properties are summarized in Tables 4 and 5. The highest efficiency is that of a cycle of the Brayton type, but it is a null cycle and any attempt to approach it for the sake of high efficiencies is at the cost of a high compression ratio and a very small value for W. If this cycle be left out of account, the best of those remaining, from every point of view, is

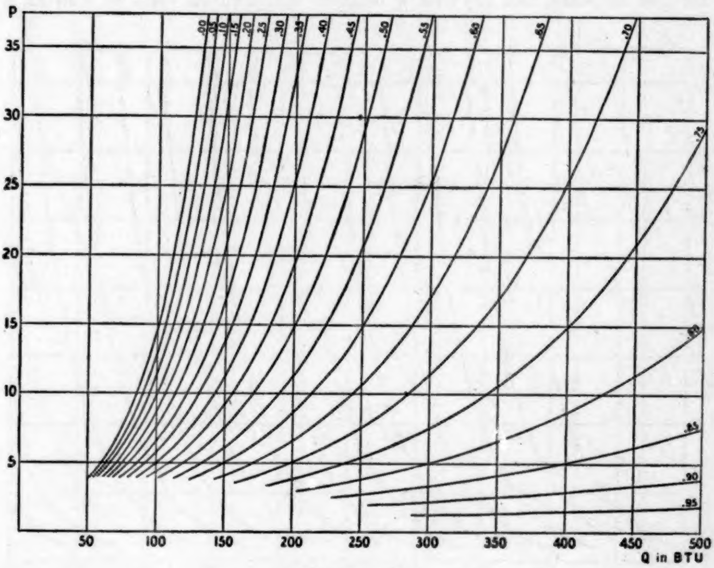


FIGURE 17.

Type C. Efficiency. For regenerative cycles. These curves, like those of Figure 14, converge toward $Q = 0, P = 1$. The line $P = 1$ corresponds to $E = 1.00$. (Cf. Figures 6 and 11.)

TABLE 4. THE EFFICIENCY CYCLES.

	P	Q	t_e	V	E	W
Type A. Brayton . . .	75.5	0	80	4000	.713	0
Type B. Isothermal. .	41.3	358	239	4000	.509	142000
Type C. Regenerative	26.2	320	400	4000	.621	154000

that regenerative cycle which corresponds to the $\bar{V}\,\bar{T}_e$ corner. Its efficiency and work are both as high as any in the tables, and its compression ratio is comparatively low. Even the isothermal cycles are usually to be preferred to any of the Brayton type on account of the much larger values of W which they involve.

The results to which this investigation has led are: (1) that of the three types considered, the regenerative cycle is, theoretically, the best for gas turbines, and (2) that a definite compression ratio as well as a

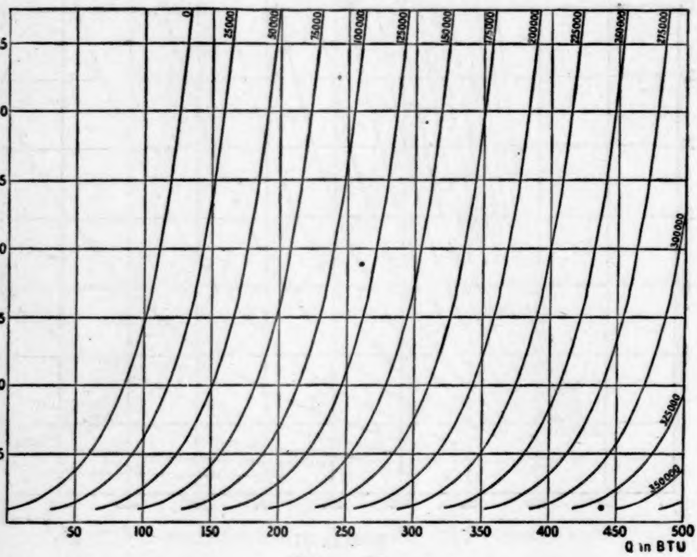


FIGURE 18.

Type C. Work. For regenerative cycles. (Cf. Figures 7 and 12.)

TABLE 5. THE WORK CYCLES.

	P	Q	t_e	V	E	W
Type A. Brayton . . .	26.2	195	400	4000	.610	92000
Type B. Isothermal .	26.2	396	400	4000	.502	154000
Type C. Regenerative	26.2	320	400	4000	.621	154000

definite consumption of fuel is determined by the conditions imposed, and should not be exceeded.

Many other interesting cyclic properties can be studied by means of the $P\,Q$ plane. As an example, the curves of highest temperature (T_d) have been plotted for types A, B, and C, and are shown in Figures 19, 20, and 21. This quantity T_d is important, not only structurally, but also because the nature of a gas is found to change very materially by reason

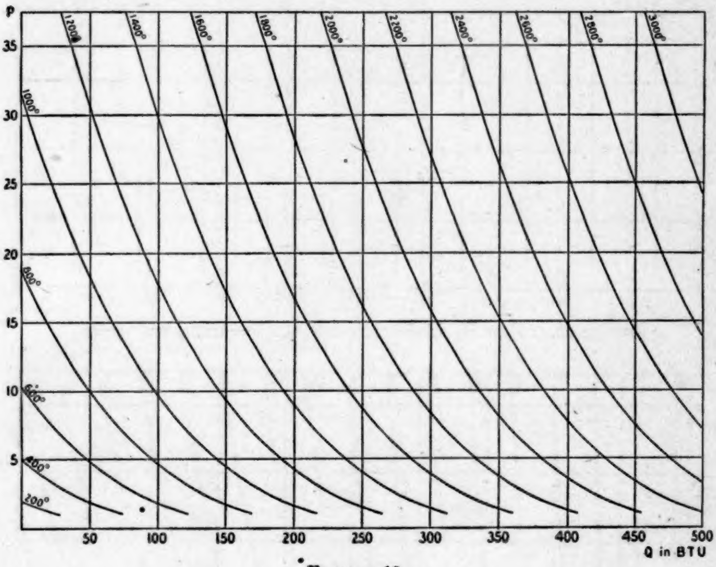


FIGURE 19.

Type A. Highest temperature. For adiabatic cycles.

of dissociation when the temperature is raised above a certain limit. On this account it may sometimes be necessary to consider, in addition to the two limiting values $\bar{T}_e$ and $\bar{V}$, a third upper limit $\bar{T}_d$, which the highest temperature should not exceed, and to rule out a part of the $\bar{V}\,\bar{T}_e$ area on account of the new condition $T_d \leqq \bar{T}_d$. In all three cases the part of the area surrounding the $\bar{V}\,\bar{T}_e$ corner would be the first to become unavailable on this account, so that the simple results obtained above would have to be somewhat modified. The graphical methods of this paper would, however, lead directly to the desired results in

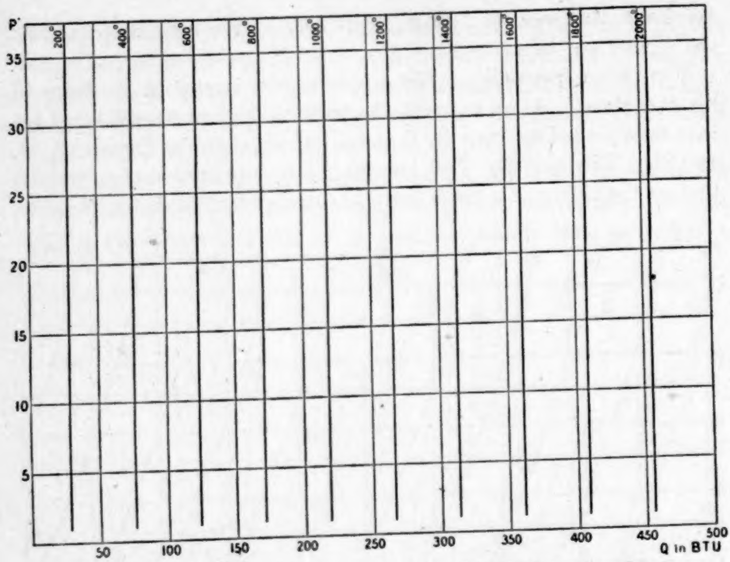


FIGURE 20. *Type B. Highest temperature.* **For isothermal cycles.**

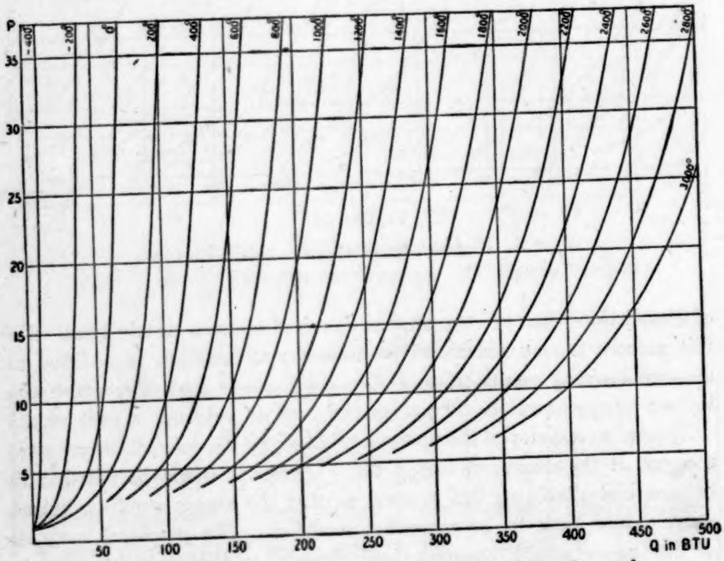
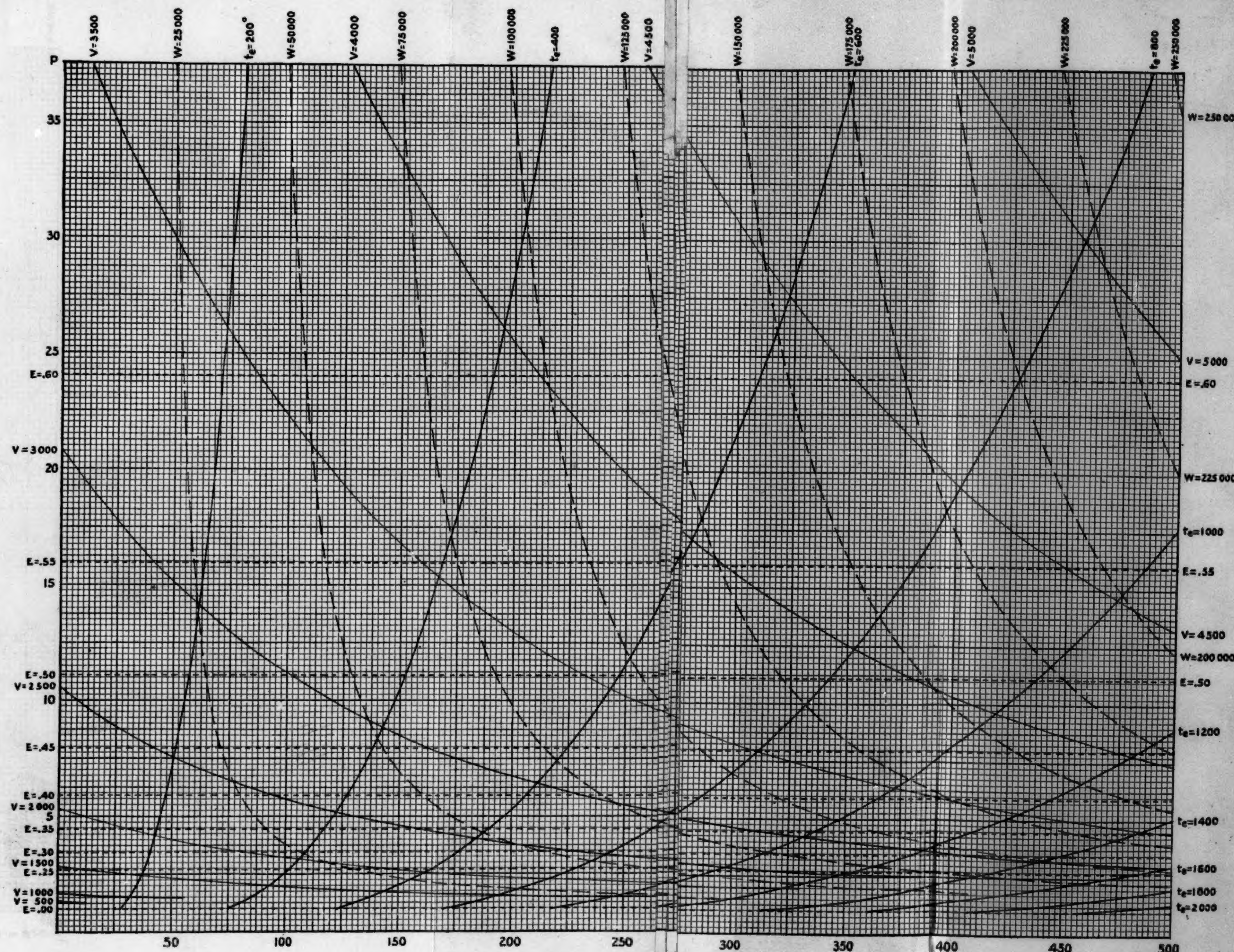
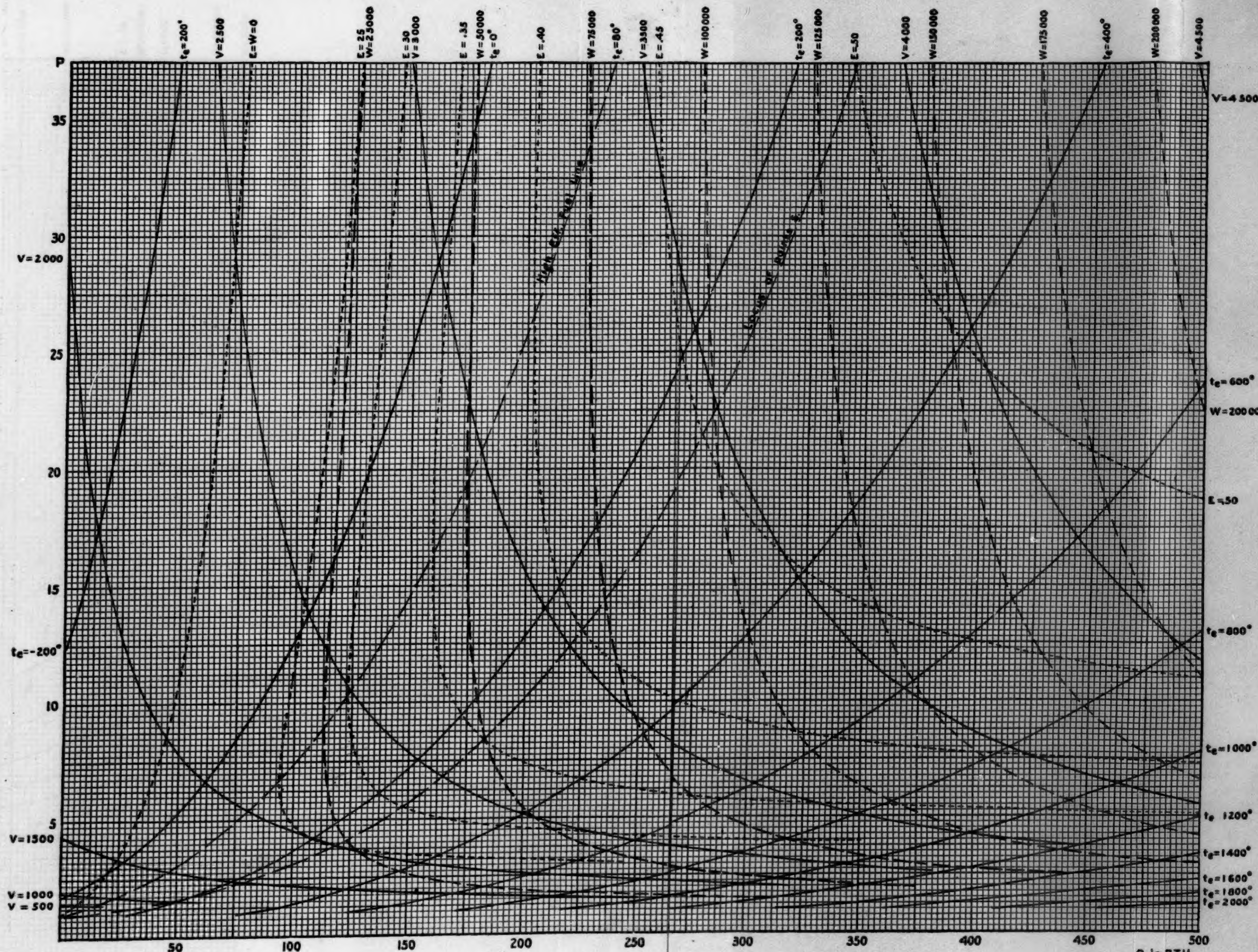


FIGURE 21. *Type C. Highest temperature.* **For regenerative cycles.**

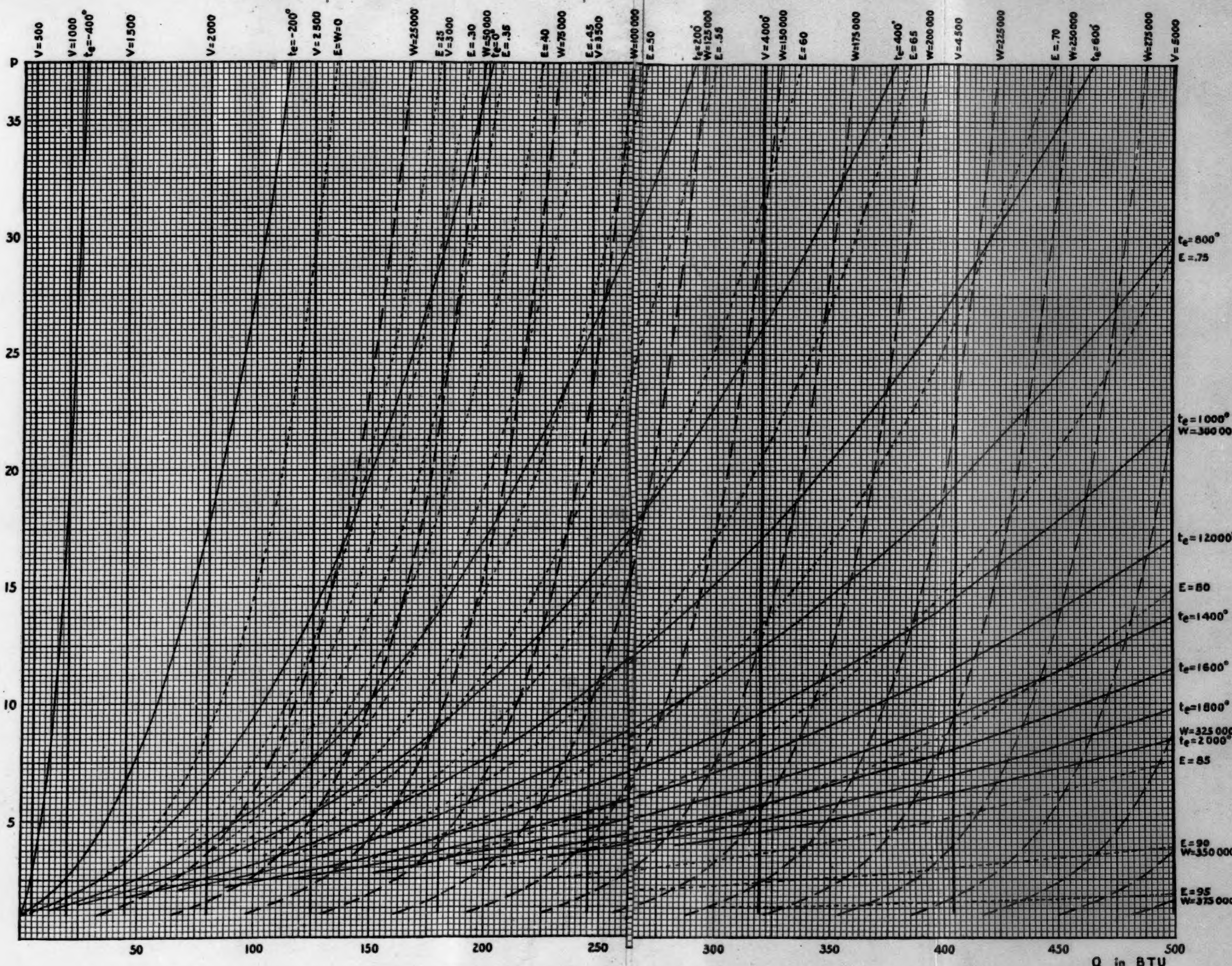
any given case, and probably also to a general theory, so that a prolongation of the present investigation is unnecessary. It is hoped that these methods will be found applicable to a great variety of problems in cyclic analysis, and that they will help in the development of this interesting subject.



TYPE A. Working Chart. Adiabatic Compression.



TYPE B. WORKING CHART. ISOTHERMAL COMPRESSION.



TYPE C. WORKING CHART. ISOTHERMAL COMPRESSION WITH REGENERATION.